MATH 146 Fall 2019
Written HW 2

Please show all of your work in a NEAT and ORGANIZED fashion.

1. If an initial amount P of money (the **principal** amount) is invested at an interest rate r compounded n times per year, the value of the investment after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

If we let $n \to \infty$, then we say the interest "compounds continuously". Use l'Hôpital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = Pe^{rt}$$

2. Estimate the numerical value of $\int_0^\infty e^{-x^2}dx$ by writing it as the sum of $\int_0^2 e^{-x^2}dx$ and $\int_2^\infty e^{-x^2}dx$. Approximate the first integral by using Simpson's Rule with $n = 8$, and show that the second integral is smaller than $\int_2^\infty e^{-2x}dx$, which is less than 0.01.

(For reference, the true value of $\int_0^\infty e^{-x^2}dx$ is $\dfrac{\sqrt{\pi}}{2} \approx 0.886227$.)

3. Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of $f(t)$ is the function $F(s)$ defined by the integral

$$F(s) := \int_0^\infty e^{-st}f(t)dt$$

The domain of $F(s)$ is all the values of s for which the integral exists. (The Laplace transform is useful in solving certain differential equations.)

(a) Determine the Laplace transform of the function $f(t) = 1$, and find the domain of the Laplace transform.

(b) Determine the Laplace transform of the function $f(t) = e^t$, and find the domain of the Laplace transform.

4. The Fibonacci sequence is the sequence given by $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$. Each term is the sum of the two preceding terms; i.e., $f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad (\text{for } n \geq 3)$.

Assume that $\displaystyle\lim_{n\to\infty} \frac{f_n}{f_{n-1}}$ exists. Find the value of the limit.

HINT: Let $x = \displaystyle\lim_{n\to\infty} \frac{f_n}{f_{n-1}}$. Write a quadratic equation in x, using the recursive definition $f_n = f_{n-1} + f_{n-2}$. Your answer will be a very special number!

5. (a) Write the decimal representation of a number $0.d_1d_2d_3\ldots$ (where the digit d_i is one of the numbers $0, 1, 2, \ldots, 9$) as a series, and show that this series always converges.

(b) Prove that $0.\overline{9} = 1$.

6. For each of the following statements, either (1) prove the statement is true, or (2) give a specific counterexample to show the statement is false.

(a) If $\sum_{n=1}^{\infty} a_n$ $(a_n \neq 0)$ converges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

(b) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges.

(c) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges.

(d) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms, where $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

7. The **Cantor set** is constructed as follows. We start with the closed interval $[0, 1]$ and remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. That leaves the two intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ and we remove the open middle third of each. Four intervals remain and we again remove the open middle third of each of them. We continue this process indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in $[0, 1]$ after all those intervals have been removed.

(a) Show that at the *nth* step $(n \geq 1)$, we remove an interval of length $\frac{2^{n-1}}{3^n}$. (For this problem, it is sufficient to just demonstrate a pattern using the first 3 steps.)

(b) Show that the sum of the lengths of all intervals removed is 1. (Note that the length of the interval $[0, 1]$ is also 1.)

(c) Show that, despite the result in part (b), the Cantor set contains infinitely many numbers. (*HINT:* Which numbers are guaranteed never to have been removed?)

8. Determine whether the series is convergent or divergent (and show your work).

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

(b) $\sum_{n=1}^{\infty} \frac{4n + 4^n}{4n + 8^n}$