12.2 – Permutations and Combinations

We will continue to learn about methods of counting, using the permutations and combinations formulas.

Motivation

- Suppose 4 different colored sheets of paper are arranged in a row. How many different ways are there to order the colors?

$$\underset{\text{1st sheet}}{\underline{4}}^{\text{colors}} \cdot \underset{\text{2nd}}{\underline{3}} \cdot \underset{\text{3rd}}{\underline{2}} \cdot \underset{\text{4th}}{\underline{1}} = \boxed{24}$$

Factorials

- Def: $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

  "n factorial"

  keep subtracting until you reach 1

- Ex: $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

  $1! = 1$

  $0! = 1$ (definition)

Permutations ☆Order matters☆

- Def: A permutation is an arrangement of objects in a definite order.

Example 1

- 8 runners compete in an Olympic race. In how many ways can the gold, silver, and bronze medals be awarded? P(8,3)

$\underset{\text{gold}}{\underline{8}} \cdot \underset{\text{silver}}{\underline{7}} \cdot \underset{\text{bronze}}{\underline{6}} = \boxed{336}$

- Alternate calculation:

# of arrangements of all 8 runners

divide out the 5 runners that you don't care about

$$\frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6$$

Permutations Formula

- The number of permutations of n objects chosen k at a time is

$$P(n,k) = \frac{n!}{(n-k)!}$$

$\rightarrow$ total

$\rightarrow$ leftovers

- <u>Note</u>: $P(n,0) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$

Example 2

- A committee of 16 students must select a president, vice president, secretary, and treasurer. In how many ways can this be accomplished?

$$\underset{P}{\underline{16}} \cdot \underset{VP}{\underline{15}} \cdot \underset{S}{\underline{14}} \cdot \underset{T}{\underline{13}} = \boxed{43{,}680}$$

OR

$$P(16,4) = \frac{16!}{(16-4)!} = \frac{16!}{12!} = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot \cancel{12!}}{\cancel{12!}}$$

Combinations ☆Order does not matter! ☆

- Def: A combination is a collection of objects whose order is not important.

Example 3

- You put 4 toppings on a pizza and you have 16 toppings to choose from. How many pizzas are possible? ⟶ combinations

1. The number of **permutations** is: $P(16,4) = 43,680$

2. This is an overcount! (The order of toppings does not matter.) How many ways can 4 toppings be arranged? $\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 24$

3. Divide to correct for the overcount: $\frac{43,680}{24} = \boxed{1,820}$

Combinations Formula

- The number of combinations of n objects chosen k at a time is

$$C(n,k) = \frac{\text{\# of permutations}}{\text{\# of arrangements of } k \text{ things}} = \frac{\boxed{P(n,k)}}{k!} = \frac{n!}{(n-k)! \cdot k!}$$

- Note: $C(n,0) = 1$

Example 4

- (a) A committee of 6 people is chosen from a group of 12. How many committees are possible? $\longrightarrow$ combinations

$$C(12,6) = \frac{P(12,6)}{6!} = \frac{\underline{12}\cdot\underline{11}\cdot\underline{10}\cdot\underline{9}\cdot\underline{8}\cdot\underline{7}}{\underline{6}\cdot\underline{5}\cdot\underline{4}\cdot\underline{3}\cdot\underline{2}\cdot\underline{1}} = \boxed{924}$$

OR

$$C(12,6) = \frac{12!}{6!\cdot 6!} = \boxed{924}$$

Example 4

- (b) A committee of 6 people is chosen from 8 men and 4 women. How many committees are possible that consist of 3 men and 3 women? Multi-stage experiment: use the counting principle

$$\underbrace{C(8,3)}_{M} \cdot \underbrace{C(4,3)}_{W} = 56 \cdot 4$$

$$= \boxed{224}$$

Example 4

- (c) 4 cards are randomly chosen from a standard deck of playing cards. How many hands contain exactly 2 queens and 1 king?

$52 - 4 - 4 = 44$ cards are NOT Q or K

Counting principle: $\underbrace{C(4,2)}_{2\text{ Qs}} \cdot \underbrace{C(4,1)}_{1\text{ K}} \cdot \underbrace{C(44,1)}_{1\text{ other}}$

$= 6 \cdot 4 \cdot 44$

$= \boxed{1{,}056}$